Philosophical Aspects of Geometrical Thinking

Dr Irina STARIKOVA

Research Fellow

University of Bristol, UK

plxis@bris.ac.uk

A Resesach Proposal for the postdoctoral position at the University of San Paulo

Potential Supervisor

Professor Osvaldo Frota Pessoa Jr

Department of Philosophy, University of San Paulo

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Abstract

This project belongs to the study of the nature and development of scientific reasoning, In particular, it focuses on the use of effective theoretical instruments such as a geometric approach to an non-geometric area for example algebra. It dwells on both its epistemological and historiographical aspects but with the emphasis on the most recent scientific discoveries, for instance the latest achievements in geometric group theory. The results of this project should lead to new premises about the foundations and progress of scientific knowledge. Potentially it should have an impact on philosophical views of formal and cognitive aspects of science, the role of technologies in science and education.

Geometric approach is becoming more and more powerful tool in recent science and mathematics in particular. The main concern of this project is the epistemological role of *visual constructions* in geometrisation. Geometrisation here is understood as *an application of geometric concepts and techniques to non-geometrical problems.* The crucial component of a geometrisation is making a bridge between two different matters as well as styles of reasoning: for example algebra and geometry.

This project aims to explain how it is possible, given a purely algebraic object normally studied algebraically, to recognise the ways in which this object can be studied geometrically. On the basis of the case study from recent mathematics the project aims to demonstrate that geometrisation may involve the *change from symbolic to diagrammatic* reasoning.

General Introduction

This is an exciting opportunity for me to apply for the position of a post doctoral researcher at the Faculty of Philosophy, USP. I am enthusiastic about contributing to USP philosophy research and teaching. I recently received my Ph.D. from one of the most highly rated universities in the UK – the University of Bristol. During my doctoral research in the UK (as well as before and up to now) I have been building up my experience in publishing, editing, fund raising and teaching. Now I want to take up the challenge to bring these knowledge and skills to such a progressive and prestigious university as USP.

My area of expertise includes philosophy of science, philosophy of mathematics, philosophy of mind, logic, mathematics and cognitive psychology. This expertise has been developed in published papers and courses that I have designed in these disciplines. I have been teaching philosophy of mathematics, logic, mathematics, epistemology and cognitive psychology in the UK and Russia. Using this experience, I believe will be able to co-supervise and motivate the students at USP through giving seminars, initiating discussions and assisting in the development of teaching materials.

I have an extended networking experience and during my career, for example through editing a special issue of TOPOI in 2010, I managed to create strong international contacts. Therefore, I would be very keen on organising a high level conferences and workshops with top specialists in the area and following publications in the top philosophical journals. I am also very keen on playing my part on lively SUP philosophical society and bringing my connections with philosophy centers in Bristol and London as well as the Siberian Academy of science, where I was raised.

Professor Osvaldo Frota Pessoa Jr from the Department of Philosophy of USP is familiar with my research on the epistemic roles of cognitive representations in scientific reasoning. He read this proposal and my other related papers, and approved that my research would go well in accordance and be a valuable integrative part of the departmental overall searcher.

1 Introduction to the problem

In terms of language and means of expression, mathematics extends towards higher and higher abstraction, and definitely after Descartes, employing more of algebraic symbolism than geometric diagrams. Does it mean that the traditional geometrical thinking has been overtaken by the abstract, algebraic one?

In the 18th – 20th century mathematics was mostly focused on formalisation and axiomatisation (Weierstrass, Cauchy, Dedekind, Bolzano, Kronecker and others). Among important motivations were of course mathematical certainty and rigour.[1] For these purposes the algebraic style of thinking was perhaps more suitable, al least straightforward. Historians suggest that this attitude is rooted in the 18th century 'desire to base mathematics on algebra', 'where the arguments were valid because they were the outcome of a purely formal manipulation of symbols' (Gray 1987). A similar explanation of the anti-geometric attitude by Tappenden (2005, 2008) has been made about a computational motivation detected, for example, in Weierstrass' approach.

One of the revolutionary examples of algebraisation in mathematics, and more to that, geometry, is Klein's 1872 "Erlangen programme". Klein proposed the *group-theoretic reconstruction of geometries.* Roughly speaking, this program aimed to unify a number of geometries by associating each geometry to a group of transformations. In other words, a geometry can be classified and studied in terms of algebraic invariants.

The philosophical outcomes of the programme has been discussed extensively e.g. in (Gray 2005). However, it often falls out from attention of philosophers and historians of mathematics that there is also no less revolutionary and successful *reverse* process: geometrisation. Generally, a *geometric approach* applied to a non-geometric subject, or *geometrisation* is an adaptation of geometrically originated concepts and methods to non-geometric problems. It is more than just an application because the concepts which are applied may be modified with respect to the given problem. In addition to geometric algebra, the famous examples include the geometrisation of number theory by Minkovsky (1910) and one of the latest examples – Perelman's (2003) geometrisation of topology.

Interestingly, for the Kleinen Erlangen programme, there also exist a reverse – geometric – approach: studying and classifying algebraic groups by imposing on them geometric structure (to be explained in the next paragraph). This approach has been developed in geometric algebra, or more precisely, in geometric group theory. In the early 20th century, Dehn (1911) used hyperbolic geometry to solve the word problem in a surface group. More recently, in 1980s, Michael Gromov (1981,1987, 1993) established a new programme which turns the Kleinean view in a sense up-side-down and suggests that groups themselves are geometric objects. This new approach brought solutions to many long-waiting algebraic problems.[2]

[1]See overviews e.g. in Mancosu (2008), Corfield (2003) and more recent Ferreiros, J. and Gray, J. (eds.) (2006).

[2]See e.g. Gromov (1981) and for newer results Kleiner (2007), Shalom et al (2009), Shalen (1979), Johannson (1979), Serre (1980) and Sela (1995). For a recent overview see Gromov (2003), Bridson (1999, 2003) and Harpe (2000), Benakli, N. and Kapovich, I. (2000).

Gromov's work is breaking new ground in the study of groups: this approach actually puts a geometric structure onto the group itself. This non-evident representation can be done through the following steps. First one constructs a graph of a group (more precisely, a finitely generated group).[3] Graphs of groups were suggested by Arthur Cayley in Cambridge in 1878 and since then are called *Cayley graphs*. Second, and this is the most astonishing twist: one can look at the graph as a geometric figure (similar to as we look at a triangle or a circle) and introduce metric on graph's edges (this needs to be explained in detail). Then, given this metric, one can treat the graph as a metric space and apply the well known geometry of metric spaces. Finally, one has to prove that graphs preserve the structure of groups up to a particular precision and relate the results about graphs to their groups.[4] On the basis of this approach, groups themselves can be thought of as geometric objects – metric spaces.

The described above geometric approach is an important epistemological phenomenon of mathematical practice. Instead of traditional abstraction as in the Kleinean programme, it begons with a decrease of abstraction (Cayley graphs of generated groups are less abstract objects than groups as such). Then, on the second step, we are dealing with a categorical shift: purely algebraic objects – Cayley graphs (as well as generated groups) – are considered as geometric objects. Namely, the elements of these algebraic objects – vertices and edges – are treated as parts of a geometric object – metric space. This allows for an application of geometric concepts, such as *metric* or *curvature*, to the Cayley graphs, and finally, groups. Observing these approach one can think of an application of geometric type of thinking to an algebraic problem. So far geometrisation has not been discussed at any length, and the aim of this research is to address this important topic. Therefore, the ***aim*** of the research is to provide an epistemological analysis of geometrisation.

2 Research questions

The following questions are going to be discussed in this project:

1. How does geometrisation work from an epistemological perspective?
2. Given a purely algebraic task, how is it possible to gain new mathematical knowledge by means of a geometrisation of that theory?
3. In particular, how can one *introduce* geometry into a purely algebraic theory?
4. Is the step from algebra to geometry solely logical or intuitive, or something else entirely (e.g. an interaction of the two)?
5. For instance, what is the epistemic role (if any) of the visual in the process of geometrisation?
6. In particular, are visualisations only auxiliary psychological aids, illustrations or can they play any explanatory or even justificatory role?
7. Is geometrisation effective because it is 'more intuitive' and 'more visual'?

[3] See Appendix.

[4] It has been shown that many of the geometric properties of Cayley graphs do not depend on the choice of generators! See e.g. Alonso (1990), Gromov (1988).

3 Literature review and methodology

Geometric vs.algebraic type of reasoning. Reliability of diagrammatic reasoning – development of representations

The questions above concern the distinction between algebraic and geometric types of thinking. This in turn involves the distinction of visual representations: symbolic notations *vs.* geometric diagrams. Kant famously distinguished between the two types of intuition that concern geometric and algebraic/arithmetical types of reasoning: the intuition of space and time. He used the term 'intuition' to refer to a distinct cognitive capacity for quasi-perceptual or imaginative acquaintance. It is close to sensory cognition in being concrete, but it is not used as empirical evidence in making a judgement. Kant notably describes it as 'pure' or synthetic *a priori*. Kant presented his argument why the use of external to mathematics objects – diagrams – preserves the purity of the geometric judgement. Giaquinto (2008) shows that Kant's argument is not strong enough, and makes an attempt to rescue it. Also, Kant's approach is constrained by the premise that our perception of the world is solely Euclidean. This problem has been discussed in detail in Friedman (2010).

However, mathematical visual means are developing towards higher reliability it seems to be possible to *develop different ways* of looking at the same mathematical concept to realise its new properties and connections. This view is not committed to Euclidean restriction: one can develop pre-empirical imaginary of non-Euclidean spaces and other 'counter-intuitive' constructions. The conception of 'cultivated' intuition was also emphasised by for example Klein, and in current literature by Feferman (2000) and Heinzmann (2007). This project suggests to explore the option that intuition is capable of developing *via* systematic reasoning, and of deepening and varying our grasp of concepts.

'Geometric' in light and strict sense

The problem of the role of diagrams in geometrisation concerns at least three common observations:

- Traditionally, geometry (Euclidean, common sense school geometry) is perceived as being appealing to visualisations.
- Some geometric theories are difficult to visualise (at least, in the traditional straightforward way).
- Some recent mathematical theories are often called to be 'geometric' by virtue of the visual elements specific for them (such as diagrammatic notations in the category theory).

To account these observations I will distinguish between two meanings of the words 'geometry' and 'geometric': the use of geometric *concepts*, *techniques* and *propositions* on the one hand, and the use of geometric *visualisations (diagrams)* on the other. It will be shown that the dichotomy-based specifications (algebraic-geometric) are somewhat relative and do not capture the richness and variety achieved by combining geometric and algebraic thinking. Finally there will be made an argument that geometrisation in the strict sense can be realised through geometrisation in the light sense. The argument will be based on the analysis of a case study introduced above: the geometrisation of groups as outlined in Gromov (1993).

The case study

Discussions of geometric reasoning traditionally address examples from Euclidean geometry, in which intuition seems to have a central place. Only a few attempts have considered the recent mathematical situation. Giaquinto (2007) and Brown (1999) give some only brief examples from relatively new subjects such as knot, graph and braid theories. In their examples visualisations play an important epistemic role but not as a part of geometrisation. Giaquinto analyses diagrammatic geometric representations and compares them to algebraic linear notations and algebraic diagram-like intermediate cases (2007, ch. 12). Gray (1987) and Tappenden (2005, 2008) give a more historical approach to cases of the 18th century mathematics, where they stress the geometric approach. Arana's (2009) paper about the advantages of geometric proofs over algebraic proofs is in the same spirit and again up to the 18th century. Manders (1999) analyses Descates' algebraisation of Euclidean geometry but does not look at reverse cases. Therefore, geometrisation is a new topic to the philosophical literature.

In contrast – and this is another novelty of the project – an investigation into the geometric approach will be presented here in a case study from recent mathematics, namely geometric group theory. It is particularly important to understand how the current highly abstract mathematics is employing geometrisation, when the latter would be normally seen rather as an ancient practice.

4 The objectives

Finally, the objectives of the research can be listed as the following:

1. To provide the background introduction, which allies classical account to the problem of geometric thinking and its related questions (the relation of the conceptual to the intuitive, the role of visualisations and the place of geometry in the foundations of mathematics).

2. To provide a discussion of geometric thinking in comparison to algebraic thinking. Give a brief introduction to geometrisation in mathematics, explaining the concepts of 'geometrisation' and 'geometric'.

3. Present the case study and open the discussion. Explain how groups, previously studied algebraically, were represented as graphs and then by defining a metric on the latter, as metric spaces. Explain the efficacy of geometrisation, namely, what mathematical progress in the domain has been achieved by this method.

4. Finally, make the general philosophical claims from the analysis of the case study.

5 The scope of this research

Regarding the questions about the roles of the visual, this project will focus more on these roles in mathematical discovery, but will shed some light on the questions of justification. The role of the visual in mathematical proofs is the topic of an ongoing discussion. I share the view (e.g. Giaquinto's) that pictures are a necessary part of some proofs, given that 'proof' cannot be reduced to written deductive strings from axioms to a conclusion. Real mathematical proofs often require far more rich and complex reasoning.

Within geometrisation, there are definitely creative moments of mathematical discovery. This research will not go into the details of those, but will only mark the strategic elements which assist discovery. Instead, the emphasis will be made on such activities as *representation* and *application*. The related issues of cognitive psychology will be also left out of this project, for this would require much more extended research.

The questions about geometry of space in the connections to the physics of real space will be generally avoided. These questions are discussed in detail in Friedman (1983) and Gray (2006). For the future development of this case study, it would be instructive to examine other relevant types of diagrams within geometric group theory (e.g. Van Kampen diagrams).

6 Expected outcomes and novelty

Therefore, the expected outcomes of this research can be stated as the following:

1. An analysis of geometrisation.
2. An analysis of a new case study from recent mathematics.
3. An analysis of possible roles, which visualisations play in geometrisation.
4. An evidence for that a change in representation (from algebraic symbols to geometric diagrams) affects the content of thought and the development of concepts. For example, the traditional view that groups are purely algebraic objects and should be studied algebraically has been refined by the new perspective: i.e. to conceive of groups as geometric objects studied by geometric methods.
5. A demonstration of the case where important results have been obtained not only in mathematical content, but also in the development of convenient cognitive representations (e.g. diagrams).

7 Appendix

Definition (A generating set). Let G be a group. Then a subset $S \subseteq G$ is called a *generating set* for the group G if every element of G can be expressed as a product of elements of S or inverses of elements of S.

Definition (A finitely generated group). A group with a specified set of generators S is called a *generated group* and is designated as (G,S). If a group has a finite set of generators, it is called a *finitely generated group.*

Definition (A Cayley graph). Let (G, S) be a finitely generated group.

Then a Cayley graph $\Gamma(G, S)$ of a group G with respect to the choice of S is a directed coloured graph, where vertices are identified with the elements of G and the directed edges of a colour s connect all possible pairs of vertices (x, sx), $x \in G$, $s \in S$.

Example 1. To make it more practical let us draw a Cayley graph for the first example, $(\mathbb{Z}, +)$. Let a generating set be $S = \{1\}$. Given that we have only one generator, all the edges in our Cayley graph will be the same colour. First, put a vertex labelled 0 for the identity element. Then apply the group operation to the identity element by multiplying it by the generator 1, which is $0+1=1$. It gives us vertex 1, and the directed edge connecting vertex 0 and the new vertex corresponding to the obtained element 1. The direction corresponds to the move from vertex 0 to vertex 1. Analogously, $0+(-1)=-1$, but in this case the edge is ***opposite*** to the direction of the move from vertex 0 to vertex -1 because we multiply the group element 0 by an *inverse* of the generator. Although we make a step to the left from 0, we direct the edge opposite to our movement (group action). As a result, the Cayley graph is an infinite chain as illustrated in the figure below:

Figure 7.1: The Cayley graph of the group $(\mathbb{Z}, \{1\})$.

Different choices of generators give different Cayley graphs. The same group $\mathbb{Z}$ with generators $\{1,2\}$ can be depicted as an infinite ladder, as in Figure 7.2:

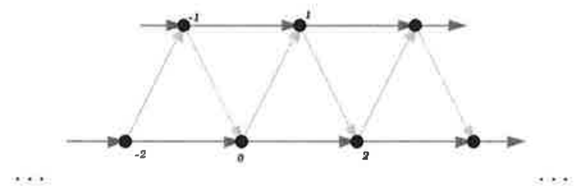

Figure 7.2: The Cayley graph of the group $(\mathbb{Z}, \{1,2\})$, where green stands for $\{1\}$ and red stands for $\{2\}$.

and $(\mathbb{Z},\{2,3\})$ in the figure below gives the graph:

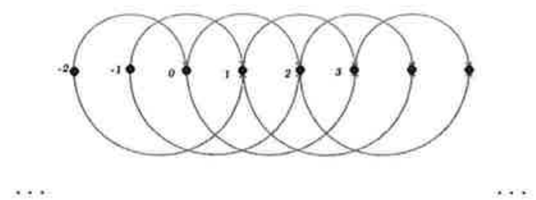

Figure 7.3: The Cayley graph of the group $(\mathbb{Z}, \{2,3\})$, where red stands for $\{2\}$ and blue stands for $\{3\}$.

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